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BENDING OF A STRETCHED PLATE CONTAINING AN ECCENTRICALLY PLATE REINFORCED HOLE OF ARBITRARY SHAPE

W. B. FRASER
Department of Applied Mathematics, University of Sydney, N.S.W. 2006, Australia

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Abstract—In this paper we consider the problem of a stretched plate containing a hole of arbitrary shape which is reinforced by thickening the plate, on one side only, in a region surrounding the hole. Due to the eccentricity of the reinforcement a bending boundary layer occurs in the neighbourhood of the junction between the plate and the reinforcement. The equations for the moments at the junction are found to be identical to those for the circular hole in Ref. [1]. The boundary layer occurring at a clamped edge of arbitrary shape is also discussed.

1. INTRODUCTION

In a previous paper [1] we solved the title problem for the case of a plate containing a circular hole, that is reinforced in an annular region surrounding the hole by thickening the plate uniformly on one side only. This plate was subjected to biaxial tension at infinity, and the bending moment induced at the junction between the plate and the reinforcement was found to depend only on the radial tension at the junction and the thicknesses of the plate and the reinforced area. This result is clearly valid for any plane stress state at infinity provided it produces a radial *tension* at the junction between the plate and reinforcement which is required for the existence of the bending boundary layer at the junction.

In this paper we show that the result of Ref. [1] holds for a hole of arbitrary shape, provided that the boundary curve of the junction between the plate and the reinforcement has a sufficiently smoothly varying curvature, and that the component of the membrane stress resultant normal to this boundary curve is a tension. A heuristic derivation of these results is given in Appendix B.

We also establish the result that, for a thin prestressed plate subjected to transverse deflections and having sufficiently smooth boundary curves, the membrane stress resultants and displacements are not affected by the transverse displacement in the first approximation.

The paper concludes with a discussion of the boundary layer that arises at a clamped boundary curve of arbitrary shape.

2. THE KÁRMÁN PLATE EQUATIONS

The dimensionless form of the Kármán large deflection plate equations, expressed in two-dimensional cartesian tensor notation and referred to dimensionless cartesian axes x_α, $\alpha = 1, 2$ lying in the middle surface of the undeformed plate, are set out in this section. Greek subscripts take values 1, 2 and the summation convention applies for repeated subscripts. The definitions of the variables used, and the relations between the dimensional and dimensionless variables are listed in the Appendix A.

The transverse equilibrium, and stress function compatibility equations are respectively

$$\epsilon^2\nabla^4 w = L(w, f) - q = N_{\alpha\beta}w_{,\alpha\beta} - q, \tag{1}$$

and

$$\nabla^4 f = -\epsilon^2 L(w, w) = -\epsilon^2 e_{\alpha\gamma}e_{\beta\lambda}w_{,\alpha\lambda}w_{,\beta\gamma}, \tag{2}$$

where

$$(\ )_{,\alpha} = \partial(\ )/\partial x_\alpha, \qquad \nabla^4(\ ) = (\ )_{,\alpha\alpha\beta\beta},$$

and

$$e_{11} = e_{22} = 0, \qquad e_{12} = -e_{21} = 1.$$

The parameter ϵ is defined by

$$\epsilon^2 = \frac{D}{PL^2} \tag{3}$$

where D is the flexural stiffness of the plate, L is some typical length dimension (e.g. the size of the reinforced hole in the case of the title problem, Fig. 2) and P is the magnitude of a (tension) stress resultant at some convenient point in the plate (e.g. the uniform tension at infinity in the case of a plate with a hole subjected to uniaxial tension).

The dimensionless transverse load

$$q = \frac{\gamma L^2 \bar{q}}{hP}, \tag{4}$$

where $\gamma = [6(1-\nu^2)]^{1/2}$, h is the plate thickness, and $\bar{q}$ is the actual dimensional load, is assumed to be of order one. If this is the case, then the boundary layer equations are, to order ϵ, independent of q.

Membrane stress resultants are related to the stress function f by

$$N_{\alpha\beta} = e_{\alpha\gamma} e_{\beta\lambda} f_{,\gamma\lambda} \tag{5}$$

and the moments, in terms of the transverse displacement, are

$$M_{\alpha\beta} = -\epsilon^2[(1-\nu)w_{,\alpha\beta} + \nu w_{,\gamma\gamma}\delta_{\alpha\beta}], \tag{6}$$

where $\delta_{11} = \delta_{22} = 1$, $\delta_{12} = \delta_{21} = 0$ is the Kronecker delta. The transverse shear stress resultant is given by

$$Q_\alpha = M_{\alpha\beta,\beta} = -\epsilon^2 w_{,\alpha\beta\beta}. \tag{7}$$

The strain-displacement-stress resultant relations are

$$E_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha} + \epsilon^2 w_{,\alpha} w_{,\beta}) = (1+\nu)N_{\alpha\beta} - \nu N_{\gamma\gamma}\delta_{\alpha\beta}. \tag{8}$$

On the boundary of the plate we prescribe

$$\text{either} \quad N_{\alpha\beta} n_\beta \quad \text{or} \quad u_\alpha, \tag{9}$$

$$\text{either} \quad M_n = e_{\beta\alpha} M_{\gamma\beta} n_\gamma t_\alpha \quad \text{or} \quad w_{,\alpha} n_\alpha, \tag{10a}$$

and

$$\text{either} \quad Q_{\text{eff}} = Q_\alpha n_\alpha + N_{\alpha\beta} n_\beta w_{,\alpha} - \frac{\partial M_t}{\partial s} \quad \text{or} \quad w, \tag{10b}$$

where

$$M_t = e_{\beta\alpha} M_{\gamma\beta} n_\gamma n_\alpha,$$

n_α, t_α are the normal and tangent to the boundary curve in the directions shown in Fig. 1, and s is distance measured along the boundary.

Dimensionless variables are defined in such a way that, over most of the plate, they and their derivatives are of order one. Further, the parameter ϵ is assumed to be very small ($\epsilon \ll 1$); that is the plate is thin, so that D/L^2 is small, and subjected to a fairly large tension so that P is large. Thus, nearly everywhere the left side of eqn (1), the right side of eqn (2) and the bending moments (6) are negligible and over most of its area the plate behaves as a non-uniformly stretched membrane, $N_{\alpha\beta} w_{,\alpha\beta} = q$, where $N_{\alpha\beta}$ is determined from the solution of the plane stress equation $\nabla^4 f = 0$. However, if the left side of eqn (1) is set equal to zero, its order is reduced by two, and one of the boundary conditions (10) must be dropped. We expect therefore that, near the boundaries, bending

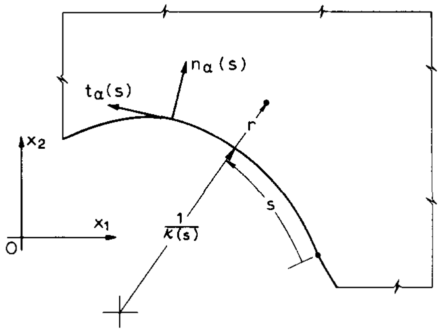

Fig. 1. Boundary geometry and coordinates.

boundary layers will occur in which w is changing very rapidly so that derivatives on the left of eqn (1) become of order ϵ^{-2} or larger and these terms and the bending moments (6) cannot be neglected in the boundary layers. We use the method of matched asymptotic expansions [2] to find a solution which is asymptotically valid as $\epsilon \to 0$.

3. PERTURBATION EQUATIONS

3.1 *Equations for the outer expansions*

Outside the boundary layers we expand all variables as power series in ϵ. For example,

$$u_\alpha = \sum_{n=0}^{\infty} \epsilon^n u_\alpha^{(n)}(x_1, x_2). \tag{11}$$

On substituting these series expansions into the equations of Section 2 and equating the coefficients of powers of ϵ on each side of the resulting equations we obtain the following system of equations:

The equations for the order one terms are

$$N_{\alpha\beta}^{(0)} w_{,\alpha\beta}^{(0)} = q, \qquad M_{\alpha\beta}^{(0)} = 0, \qquad Q_\alpha^{(0)} = 0, \tag{12a}$$

$$\left.\begin{aligned} \nabla^4 f^{(0)} = 0, \qquad N_{\alpha\beta}^{(0)} = e_{\alpha\gamma} e_{\beta\lambda} f_{,\gamma\lambda}^{(0)}, \\ E_{\alpha\beta}^{(0)} = \tfrac{1}{2}(u_{\alpha,\beta}^{(0)} + u_{\beta,\alpha}^{(0)}) = (1+\nu) N_{\alpha\beta}^{(0)} - \nu N_{\gamma\gamma}^{(0)} \delta_{\alpha\beta}, \end{aligned}\right\} \tag{12b}$$

for the order ϵ terms

$$N_{\alpha\beta}^{(0)} w_{,\alpha\beta}^{(1)} = -N_{\alpha\beta}^{(1)} w_{,\alpha\beta}^{(0)}, \qquad M_{\alpha\beta}^{(1)} = 0, \qquad Q_\alpha^{(1)} = 0, \tag{13a}$$

$$\left.\begin{aligned} \nabla^4 f^{(1)} = 0, \qquad N_{\alpha\beta}^{(1)} = e_{\alpha\gamma} e_{\beta\lambda} f_{,\gamma\lambda}^{(1)}, \\ E_{\alpha\beta}^{(1)} = \tfrac{1}{2}(u_{\alpha,\beta}^{(1)} + u_{\beta,\alpha}^{(1)}) = (1+\nu) N_{\alpha\beta}^{(1)} - \nu N_{\gamma\gamma}^{(1)} \delta_{\alpha\beta}, \end{aligned}\right\} \tag{13b}$$

and so on.

The solutions of these equations must be matched to the boundary layer solutions through a matching condition to be described later. We note that eqns (12b), which are just the equations of plane stress, do not involve the transverse displacement w, so that if the order one matching condition on $f^{(0)}$, $N_{\alpha\beta}^{(0)}$ and $u_\alpha^{(0)}$ does not involve w these solutions will be independent of w. The same remark applies to eqns (13b), and it is only at order ϵ^2 that the equations for the membrane stress resultants and displacements involve w.

3.2 *The boundary layer equations*

In order to derive the boundary layer equations, the equations of section 2 must first be expressed in terms of coordinates appropriate to the shape of the boundary. Let S, R be distances measured along and normal to the boundary respectively and let $\rho(S)$ be the radius of curvature of the boundary at S. We introduce dimensionless coordinates and curvature (see Fig. 1)

$$r = R/L, \qquad s = S/L, \qquad \kappa(s) = (\rho/L)^{-1}, \tag{14}$$

and unit normal and tangent vectors to the boundary $n_\alpha(s)$, $t_\alpha(s)$ so that $x_\alpha = \bar{x}_\alpha(s) + rn_\alpha(s)$, where $\bar{x}_\alpha(s)$ are the coordinates of points on the boundary, and

$$t_\alpha = \frac{\mathrm{d}\bar{x}_\alpha}{\mathrm{d}s} = e_{\beta\alpha}n_\beta, \qquad \frac{\mathrm{d}n_\alpha}{\mathrm{d}s} = \kappa(s)t_\alpha, \qquad \frac{\mathrm{d}t_\alpha}{\mathrm{d}s} = -\kappa(s)n_\alpha. \tag{15}$$

The operator $\partial/\partial x_\alpha$ becomes

$$\frac{\partial}{\partial x_\alpha} = n_\alpha\frac{\partial}{\partial r} + t_\alpha(1+\kappa r)^{-1}\frac{\partial}{\partial s} \tag{16}$$

and hence

$$\nabla^4 \equiv \left[\frac{\partial^2}{\partial r^2} + (1+\kappa r)^{-1}\kappa\frac{\partial}{\partial r} + (1+\kappa r)^{-2}\frac{\partial^2}{\partial s^2} - (1+\kappa r)^{-3}r\frac{\mathrm{d}\kappa}{\mathrm{d}s}\frac{\partial}{\partial s}\right]^2, \tag{17}$$

and we must henceforth consider only boundaries whose curvature has a continuous second derivative $\mathrm{d}^2\kappa/\mathrm{d}s^2$.

The base vectors for the (r, s) coordinate system are n_α, t_α and the components of the displacement, moment and stress resultants, etc. relative to this coordinate system are

$$\left.\begin{aligned} u_r = u_\alpha n_\alpha, \qquad u_s = u_\alpha t_\alpha, \qquad \\ M_{rr} = M_{\alpha\beta}n_\alpha n_\beta, \qquad M_{rs} = M_{\alpha\beta}n_\alpha t_\beta, \qquad M_{ss} = M_{\alpha\beta}t_\alpha t_\beta, \end{aligned}\right\} \tag{18}$$

etc.

Using the above results the equations of Section 2 may be expressed in terms of the coordinates (r, s). As some of the expressions are fairly lengthy we shall not give them here. Rather, we shall go directly to the boundary layer equations by introducing the stretched normal coordinate

$$\eta = r/\epsilon. \tag{19}$$

In terms of the boundary layer coordinates (η, s) the operators ∇^4 and L, for example, become

$$\left.\begin{aligned} \nabla^4_{BL} &= \epsilon^{-4}\frac{\partial^4}{\partial\eta^4} + \epsilon^{-3}2\kappa\frac{\partial^3}{\partial\eta^3} + O(\epsilon^{-2}), \\ L_{BL}(A, B) &= \epsilon^{-3}\kappa\left(\frac{\partial^2 A}{\partial\eta^2}\frac{\partial B}{\partial\eta} + \frac{\partial^2 B}{\partial\eta^2}\frac{\partial A}{\partial\eta}\right) + O(\epsilon^{-2}), \end{aligned}\right\} \tag{20}$$

and we have imposed a further restriction on the curvature, namely, that $\kappa(s)$ and its derivatives are of order one.

Having expressed the equations of Section 2 in terms of these boundary layer coordinates, we substitute expansions of the form

$$w = \sum_{n=0}^{\infty} \epsilon^n \hat{w}^{(n)}(\eta, s), \tag{21}$$

and equate the coefficients of powers of ϵ on each side of the resulting equations to obtain, finally, the following system of boundary layer equations. Only those equations needed to determine the leading term in the perturbation expansions are given.

From the equation of transverse equilibrium (1) we obtain

$$\frac{\partial^4\hat{w}^{(0)}}{\partial\eta^4} - \hat{N}^{(0)}_{rr}\frac{\partial^2\hat{w}^{(0)}}{\partial\eta^2} = 0, \tag{22a}$$

$$\frac{\partial^4\hat{w}^{(1)}}{\partial\eta^4} - \hat{N}^{(0)}_{rr}\frac{\partial^2\hat{w}^{(1)}}{\partial\eta^2} = -2\kappa\frac{\partial^3\hat{w}^{(0)}}{\partial\eta^3} + \hat{N}^{(1)}_{rr}\frac{\partial^2\hat{w}^{(0)}}{\partial\eta^2} + 2\hat{N}^{(0)}_{rs}\frac{\partial^2\hat{w}^{(0)}}{\partial\eta\partial s} + \kappa\hat{N}^{(0)}_{ss}\frac{\partial\hat{w}^{(0)}}{\partial\eta}, \tag{22b}$$

and from the stress function compatibility eqn (2),

$$\left.\begin{aligned} &\frac{\partial^4 \hat{f}^{(0)}}{\partial \eta^4} = 0, \quad \frac{\partial^4 \hat{f}^{(1)}}{\partial \eta^4} = -2\kappa \frac{\partial^3 \hat{f}^{(0)}}{\partial \eta^3}, \\ &\frac{\partial^4 \hat{f}^{(2)}}{\partial \eta^4} = -2\kappa \frac{\partial^3 \hat{f}^{(1)}}{\partial \eta^2} + 2\kappa^2 \eta \frac{\partial^3 \hat{f}^{(0)}}{\partial \eta^3} - 2\frac{\partial^4 \hat{f}^{(0)}}{\partial \eta^2 \partial s^2}. \end{aligned}\right\} \tag{23}$$

From the stress resultant-stress function relation (5),

$$0 = \frac{\partial \hat{f}^{(0)}}{\partial \eta}, \qquad \hat{N}_{rr}^{(0)} = \kappa \frac{\partial \hat{f}^{(1)}}{\partial \eta} + \frac{\partial^2 \hat{f}^{(0)}}{\partial s^2}, \tag{24a,b}$$

$$0 = \frac{\partial^2 \hat{f}^{(0)}}{\partial \eta \partial s}, \qquad \hat{N}_{rs}^{(0)} = -\frac{\partial^2 \hat{f}^{(1)}}{\partial \eta \partial s} + \kappa \frac{\partial \hat{f}^{(0)}}{\partial s}, \tag{25a,b}$$

$$0 = \frac{\partial^2 \hat{f}^{(1)}}{\partial \eta^2}, \qquad \hat{N}_{ss}^{(0)} = \frac{\partial^2 \hat{f}^{(2)}}{\partial \eta^2}. \tag{26a,b}$$

From the moment curvature eqn (6)

$$\hat{M}_{rr}^{(0)} = -\frac{\partial^2 \hat{w}^{(0)}}{\partial \eta^2}, \qquad \hat{M}_{rs}^{(0)} = 0, \qquad \hat{M}_{ss}^{(0)} = -\nu \frac{\partial^2 \hat{w}^{(0)}}{\partial \eta^2}, \tag{27a}$$

$$\left.\begin{aligned} &\hat{M}_{rr}^{(1)} = -\frac{\partial^2 \hat{w}^{(1)}}{\partial \eta^2} - \nu\kappa \frac{\partial \hat{w}^{(0)}}{\partial \eta}, \qquad \hat{M}_{rs}^{(1)} = -(1-\nu)\frac{\partial^2 \hat{w}^{(0)}}{\partial \eta \partial s}, \\ &\hat{M}_{ss}^{(1)} = -\nu \frac{\partial^2 \hat{w}^{(1)}}{\partial \eta^2} - \kappa \frac{\partial \hat{w}^{(0)}}{\partial \eta}, \end{aligned}\right\} \tag{27b}$$

and from the strain-displacement-stress resultant relations (8) we obtain

$$\frac{\partial \hat{u}_r^{(0)}}{\partial \eta} = 0, \qquad \hat{E}_{rr}^{(0)} = \frac{\partial \hat{u}_r^{(1)}}{\partial \eta} + \left(\frac{\partial \hat{w}^{(0)}}{\partial \eta}\right)^2 = \hat{N}_{rr}^{(0)} - \nu \hat{N}_{ss}^{(0)}, \tag{28a,b}$$

$$\hat{E}_{rs}^{(0)} = \frac{\partial \hat{u}_s^{(0)}}{\partial s} + \kappa \hat{u}_r^{(0)} = \hat{N}_{ss}^{(0)} - \nu \hat{N}_{rr}^{(0)}, \tag{29}$$

$$\frac{\partial \hat{u}_s^{(0)}}{\partial \eta} = 0, \qquad \hat{E}_{ss}^{(0)} = \frac{1}{2}\left\{\frac{\partial \hat{u}_s^{(1)}}{\partial \eta} + \frac{\partial \hat{u}_r^{(0)}}{\partial s} - \kappa \hat{u}_s^{(0)}\right\} = (1+\nu)\hat{N}_{rs}^{(0)}. \tag{30a,b}$$

Notice that the transverse load q, which we have assumed to be of order one, is not involved in any of the above equations.

The solutions of the boundary layer equations must satisfy the boundary conditions (9) and (10) appropriate to the problem under consideration. Thus, in terms of the boundary layer coordinates, and assuming the boundary conditions have been expanded in powers of ϵ, for the order one solution we must prescribe at $\eta = 0$, all s

either $(\hat{N}_{rr}^{(0)}, \hat{N}_{rs}^{(0)})$ or $(\hat{u}_r^{(0)}, \hat{u}_s^{(0)})$ or $(\hat{N}_{rr}^{(0)}, \hat{u}_s^{(0)})$ or $(\hat{u}_r^{(0)}, \hat{N}_{rs}^{(0)})$, (31)

either $\hat{M}_{rr}^{(0)}$ or $\dfrac{\partial \hat{w}^{(0)}}{\partial \eta}$, (32)

and either $\hat{Q}_{eff}^{(0)}$ or $\hat{w}^{(0)}$, (33)

where

$$\hat{Q}_{eff}^{(0)} = -\frac{\partial^3 \hat{w}^{(1)}}{\partial \eta^3} + \hat{N}_{rr}^{(0)} \frac{\partial \hat{w}^{(1)}}{\partial \eta} + \hat{N}_{rr}^{(1)} \frac{\partial \hat{w}^{(0)}}{\partial \eta} + \hat{N}_{rs}^{(0)} \frac{\partial \hat{w}^{(0)}}{\partial s} - \kappa \frac{\partial^2 \hat{w}^{(0)}}{\partial \eta^2}.$$

4. THE ORDER ONE MEMBRANE STRESS RESULTANTS AND DISPLACEMENTS

First we establish that, provided the prescription of the boundary conditions (31) does not involve the transverse displacement at the boundary, the order one approximation to the membrane stress resultants and displacements is found from the solution of the plane stress problem obtained by setting $\epsilon = 0$ in the equations of Section 2.

We note that eqns (12b) for the order one outer solution for membrane stress resultants and displacements are independent of w. We now show that these stress resultants and displacements are constant across the boundary layer, a result similar to that of the hydrodynamic boundary layer flow over a curved surface[3], and that the boundary conditions (31) can be applied directly to the outer solution.

From eqns (24a), (25a) and (26a) we obtain

$$\hat{f}^{(0)}(\eta, s) = \phi_0(s), \qquad \hat{f}^{(1)}(\eta, s) = \phi_1(s) + \psi_1(s)\eta,$$

and substituting this result into eqns (24b) and (25b)

$$\hat{N}^{(0)}_{rr} = \kappa(s)\psi_1(s) + \phi''_0(s), \qquad \hat{N}^{(0)}_{rs} = -\psi'_1(s) + \kappa(s)\phi'_0(s),$$

where $(\ )' = \mathrm{d}(\ )/\mathrm{d}s$. From eqns (28a) and (30a) we see that $\hat{u}_r^{(0)}$ and $\hat{u}_s^{(0)}$ are also independent of η, and so from (29) $\hat{N}^{(0)}_{ss}$ is independent of η. Thus $\hat{N}^{(0)}_{rr}$, $\hat{N}^{(0)}_{ea}$, $\hat{N}^{(0)}_{ss}$, $\hat{u}_r^{(0)}$, $\hat{u}_s^{(0)}$ are independent of η in the boundary layer and so must take on their values at the boundary $\eta = 0$.

The essence of matching the inner and outer expansions is that the behaviour of the inner expansion as $\eta \to \infty$ and the outer expansion as $r \to 0$ be in agreement (see Ref. [2], p. 11 ff.). For the order one terms, since the boundary layer solutions are independent of y, matching gives

$$\hat{N}^{(0)}_{rr}(s) = N^{(0)}_{\alpha\beta}(\bar{x}_1, \bar{x}_2)n_\alpha n_\beta, \qquad \hat{N}^{(0)}_{rs}(s) = N^{(0)}_{\alpha\beta}(\bar{x}_1, \bar{x}_2)n_\alpha t_\beta,$$

$$\hat{N}^{(0)}_{ss}(s) = N^{(0)}_{\alpha\beta}(\bar{x}_1, \bar{x}_2)t_\alpha t_\beta,$$

$$\hat{u}_r^{(0)}(s) = u_\alpha^{(0)}(\bar{x}_1, \bar{x}_2)n_\alpha, \qquad \hat{u}_s^{(0)}(s) = u_\alpha^{(0)}(\bar{x}_1, \bar{x}_2)t_\alpha,$$

where $\bar{x}_\alpha(s)$, $\alpha = 1, 2$ are the coordinates of points on the boundary. Thus the boundary conditions (31) can be applied directly to the outer solution, and the result stated in the first paragraph of this section is established.

5. THE PLATE WITH A PLATE REINFORCED HOLE

Concider a plate containing a hole of arbitrary shape which is reinforced by thickening the plate, on one side only, in a region surrounding the hole as shown in Fig. 2. Let C designate the boundary between the reinforced region D_1 and the rest of the plate D_2. The dimensions of the region D_1 compared with its thickness H and the region D_2 compared with its thickness h $(0 \leq \lambda = h/H \leq 1)$ are assumed to be such that Kármán large deflection plate theory is applicable in both regions. The structure is assumed to be loaded by in plane tensile stresses at its edges and $q = 0$. The plates D_1, D_2 are assumed to remain flat outside the boundary layer near C.

The details of the solution of this problem are the same as they were for the circular hole problem[1] and only a brief statement will be given here. To order one, the continuity conditions on the membrane stress resultants and displacements across C are independent of the transverse displacement w (see eqns (35) and (37) of Ref. [1]) and thus the membrane stress resultants and displacements are obtained by solving the plane stress problem for the system as though the reinforcing were symmetrical with respect to the plate middle surface. A bending boundary layer will exist near C, provided this plane stress solution is such that the normal stress resultant $N^{(0)}_{rr}$ (which is continuous across C) is positive everywhere on C.

The outer solution for the order one transverse displacement is, in the plate D_2, $w^{(0)} = 0$, and in the reinforcement D_1, $w^{(0)} = \text{constant}$, the constant being determined by matching with the boundary layer solution. We now note that the boundary layer eqns (31a), (33) of Ref. [1] are identical with eqns (22) and (27) of this paper if we set $(y, \theta) = (\eta, s)$ in the former equations. (The factor β which multiplies some of the terms in the equations of Ref. [1] allows for the fact that all variables are normalized with respect to the plate thickness h so that in the plate $\beta = 1$ and in the

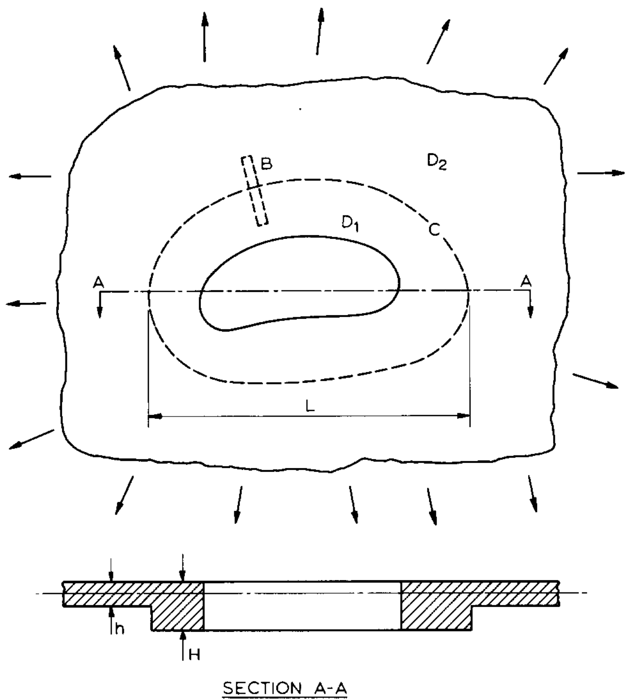

Fig. 2. Plate with an eccentrically plate reinforced hole of arbitrary shape.

reinforcement $\beta = \lambda$.) The boundary conditions are also as given in Ref. [1] and the solution for the transverse order one displacement obtained for the circular hole in Ref. [1] and the solution for the arbitrary hole are identical.

Thus, to order one, and outside the boundary layer the reinforcing plate D_1 has a constant transverse displacement (in terms of dimensional variables)

$$\bar{w} \sim \tfrac{1}{2}(H-h)$$

which brings its middle surface into the plane of the plate middle surface.

In the boundary layer, the plane stress solution stresses are increased in the upper surface of the reinforcement and the lower surface of the plate due to the bending moment induced by the eccentricity of the reinforcement. These moments and stresses have their maximum values at the boundary C, where in terms of the physical, dimensional variables of the problem they are (see eqns (63) and (64) of Ref. [1]):

In the plate;

$$\bar{M}_{rr}(S) \approx -\lambda^{3/2}(1+\lambda^{3/2})^{-1}\tfrac{1}{2}(H-h)\bar{n}(S), \tag{34a}$$

$$h\sigma/\bar{n}(S) \approx 3\lambda^{1/2}(1-\lambda)(1+\lambda^{3/2})^{-1}+1, \tag{34b}$$

In the reinforcement;

$$\bar{M}_{rr}(S) \approx (1+\lambda^{3/2})^{-1}\tfrac{1}{2}(H-h)\bar{n}(S), \tag{35a}$$

$$h\sigma/\bar{n}(S) \approx 3\lambda(1-\lambda)(1+\lambda^{3/2})^{-1}+1, \tag{35b}$$

where $\bar{n}(S)$ is the component of the stress resultant normal to C, which must be positive for all S for these results to be valid, and σ is the extreme fibre stress normal to C.

The stress quotients (34b) and (35b) are shown as functions of λ in the graphs in Fig. 3. The severest case arises if $\lambda = 0{\cdot}25$ when the extreme fibre stress in the lower surface of the plate is exactly twice the stress that would occur in the symmetrically reinforced plate. A heuristic derivation of these results is given in Appendix B.

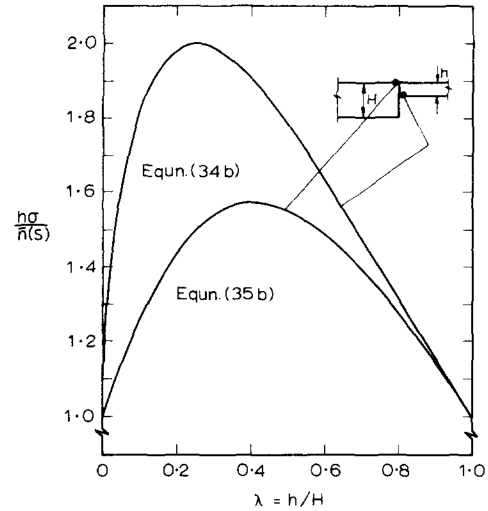

Fig. 3. Extreme fibre stress quotient versus thickness ratio λ.

6. BOUNDARY LAYER AT A CLAMPED EDGE

Apart from the reinforced hole problem described in Section 5, the most interesting boundary layer is the one which occurs at a clamped edge. This is the type of boundary layers that occur in the problem of the deflection of a radially prestressed annular plate by tilting a central rigid inclusion[4]. Whereas in the reinforced hole problem bending moments of order one arise in the boundary layer, at a clamped edge the moments are only of order ϵ.

By a clamped edge we shall mean an edge, $r = 0$, at which

$$w(0, s) = \omega(s), \qquad w_{,r}(0, s) = g(s), \tag{36}$$

where $(\ )_{,r} \equiv \partial(\ )/\partial r$, and $\omega(s)$, $g(s)$ are prescribed functions of s which are at most order one, and we assume that the transverse deflection is not involved in the prescription of the plane stress problem boundary conditions (31). In terms of the boundary layer coordinates (η, s) and expansions (21) the boundary conditions (36) become

$$\hat{w}^{(0)}(0, s) = \omega(s), \qquad \hat{w}^{(1)}(0, s) = \hat{w}^{(2)}(0, s) = \cdots = 0, \tag{37a}$$

$$\frac{\partial \hat{w}^{(0)}}{\partial \eta} = 0, \qquad \frac{\partial \hat{w}^{(1)}}{\partial \eta} = g(s), \qquad \frac{\partial \hat{w}^{(2)}}{\partial \eta} = \cdots = 0. \tag{37b}$$

We further assume that the plane stress problem has been solved so that $\hat{N}_{rr}^{(0)}(\eta, s) = N_{rr}^{(0)}(0, s) = n(s)$ is known, and that the outer solution for the transverse displacement obtained from eqns (12a), (13a), etc. can be expanded, near the boundary, as a Taylor series in r:

$$w(r, s) \approx w^{(0)}(0, s) + w_{,r}^{(0)}(0, s)r + \cdots + \epsilon w^{(1)}(0, s) + \epsilon w_{,r}^{(1)}(0, s)r + \cdots + O(\epsilon^2), \tag{38}$$

where the functions $w^{(0)}(r, s)$, $w^{(1)}(r, s)$, etc. will contain arbitrary terms which must be determined by the matching procedure.

The order one solution. The solution of eqns (22a) satisfying the boundary conditions (37) is

$$\hat{w}^{(0)}(\eta, s) = C_0(s)(n^{1/2}\eta - 1 + e^{-n^{1/2}\eta}) + \omega(s), \tag{39}$$

where $C_0(s)$ is determined by matching the inner and outer solutions.

Following Cole[3] we assume a region near $r = 0$, of overlapping validity of the inner solution (39) and the outer solution (38) and introduce an intermediate variable

$$\bar{\eta} = r/\mu(\epsilon), \quad \text{so that} \quad r = \mu\bar{\eta}, \quad \eta = (\mu/\epsilon)\bar{\eta}, \tag{40}$$

where $$\mu(\epsilon) \to 0, \qquad \mu/\epsilon \to \infty, \quad \text{as} \quad \epsilon \to 0.$$

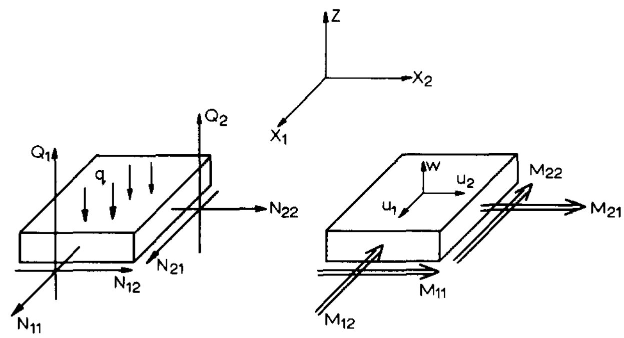

Fig. 4. Moments and stress resultants.

In terms of the intermediate variable, (38) and (39) become

$$w = w^{(0)}(0, s) + \mu w^{(0)}_{,r}(0, s)\bar{\eta} + \epsilon w^{(1)}(0, s) + O(\mu^2), \tag{41}$$

$$\hat{w}^{(0)} = C_0(s)\left\{\left(\frac{\mu}{\epsilon}\right)n^{1/2}\bar{\eta} - 1 + \exp\left(-\frac{\mu}{\epsilon}n^{1/2}\bar{\eta}\right)\right\} + \omega(s), \tag{42}$$

and the matching condition for terms of order one is

$$\lim_{\substack{\bar{\eta}\text{ fixed}}} \epsilon \to 0 \{\hat{w}^{(0)} + \epsilon\hat{w}^{(1)} + \cdots - w^{(0)} - \epsilon w^{(1)} - \cdots\} = 0. \tag{43}$$

On substituting (41) and (42) into (43) and performing the limit process we find that

$$w^{(0)}(0, s) = \omega(s) \quad \text{and} \quad C_0(s) \equiv 0, \quad \left(\text{since } \frac{\mu}{\epsilon} \to \infty \quad \text{as} \quad \epsilon \to 0\right). \tag{44a,b}$$

Thus, to order one w is constant across the boundary layer, and from (27)

$$\hat{M}^{(0)}_{rr} = \hat{M}^{(0)}_{rs} = \hat{M}^{(0)}_{ss} = 0. \tag{45}$$

Also, the order one outer solution $w^{(0)}(r, s)$ is now completely determined by the second order equation (12a) and the boundary condition (44a).

The order ϵ solution. Since $\hat{w}^{(0)}$ is independent of η, the right hand side of (22b) is zero and the solution which satisfies the boundary conditions (37) is

$$\hat{w}^{(1)} = C_1(s)\{n^{1/2}\eta - 1 + \mathrm{e}^{-n^{1/2}\eta}\} + g(s)\eta, \tag{46}$$

where $C_1(s)$ is determined by the order ϵ matching condition which is

$$\lim_{\substack{\bar{\eta}\text{ fixed}}} \epsilon \to 0 \left\{\frac{\hat{w}^{(0)} + \epsilon\hat{w}^{(1)} + \cdots - w^{(0)} - \epsilon w^{(1)} - \cdots}{\epsilon}\right\} = 0. \tag{47}$$

Writing (46) in terms of the intermediate variable $\bar{\eta}$, substituting (46), (44) and (41) into (47) and carrying out the limit process we obtain

$$C_1(s) = [w^{(0)}_{,r}(0, s) - g(s)][n(s)]^{-1/2}, \tag{48}$$

$$w^{(1)}(0, s) = C_1(s). \tag{49}$$

Equation (48) fixes $C_1(s)$ and eqn (49) provides the boundary condition on the order ϵ outer solution $w^{(1)}(r, s)$. Finally the order ϵ bending moments in the boundary layer are, from eqns (27b),

$$\hat{M}^{(1)}_{rr} = \nu^{-1}\hat{M}^{(1)}_{ss} = [g(s) - w^{(0)}_{,r}(0, s)]\,\mathrm{e}^{-n^{1/2}\eta}, \quad \hat{M}^{(1)}_{rs} = 0. \tag{50}$$

7. CONCLUDING REMARKS

We have examined the bending boundary layer that occurs at the boundary of a thin stretched plate using the Kármán large deflection plate equations. We have found that, away from its boundaries, the plate behaves as a non-uniformly stretched membrane, and that to first order the membrane stress resultants and strains are constant through the boundary layer. They are determined by the solution of the plane stress problem that would result if the transverse deflection were everywhere zero. At a clamped edge, bending moments of order $\epsilon = [D/(PL^2)]^{1/2}$ occur and they decay exponentially away from the boundary.

In the case of the title problem, bending moments of order one occur in boundary layers on either side of the thickness discontinuity between the plate and the reinforced region and the stress concentration factor at the discontinuity is doubled if the thickness ratio of the plate to the reinforcement is 1:4. Of course, in the neighbourhood of a thickness discontinuity the Kirchhoff assumption will break down and so this result must be treated with caution.

Finally, we emphasise again that the boundary layer solution is only valid if the plane stress solution gives *tensile* normal stress resultants at the boundary.

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APPENDIX A

Notation

Note: All barred variables except $\bar{x}_\alpha$ are dimensional variables.

D	$Eh^3/[12(1-\nu^2)]$, flexural stiffness
E	Young's modulus
$\bar{E}_{\alpha\beta}$	$P(Eh)^{-1}E_{\alpha\beta}$, middle surface strains
$\bar{f}$	PL^2f, Airy's stress function
h	thickness of plate
H	thickness of reinforcement
L	a representative length dimension of the plate
$\bar{M}_{\alpha\beta}$	$Ph\gamma^{-1}M_{\alpha\beta}$, bending moments
$\bar{N}_{\alpha\beta}$	$PN_{\alpha\beta}$, membrane stress resultants
$\bar{n}(S)$	$Pn(s)$, normal stress resultant at the boundary
n_α	unit vector normal to the boundary
P	representative magnitude of tensile stress resultants
$\bar{Q}_\alpha$	$Ph(L\gamma)^{-1}Q_\alpha$, transverse shear stress resultants
$\bar{q}$	$Ph(L^2\gamma)^{-1}q$, load per unit area normal to the plate surface
(R, S)	(Lr, Ls), coordinates normal to, and along, the boundary
t_α	unit vector tengential to the boundary
$\bar{u}_\alpha$	$LP(Eh)^{-1}u_\alpha$, inplane middle surface displacements
$\bar{w}$	$h\gamma^{-1}w$, transverse displacement
(X_1, X_2, Z)	(Lx_1, Lx_2, hz), dimensional cartesian coordinates
$\bar{x}_\alpha(s)$	dimensionless coordinates of the boundary curve
γ	$[6(1-\nu^2)]^{1/2}$
λ	h/H
$\kappa(s)$	$[\rho(S)/L]^{-1}$, dimensionless curvature of the plate boundary contour
$\rho(S)$	dimensional radius of curvature of the plate boundary
σ	dimensional extreme fibre stress
ν	Poisson's ratio.

APPENDIX B

Heuristic derivation of the eccentrically reinforced hole result

In this appendix we give a heuristic derivation of results (34) and (35) of Section 5. The central idea here is that in the boundary layers the behaviour of the plate and reinforcement may be obtained by considering a beam, with an appropriate thickness discontinuity, whose axis is normal to the curve C of the junction between the plate and reinforcement as shown in Fig. 2. We first derive the results for a stretched beam with a thickness discontinuity and then discuss the application of these results to the eccentrically plate reinforced hole of arbitrary shape. All the variables used in this appendix are dimensional variables.

The argument that follows is due to Prof. W. H. Wittrick and I am indebted to him for drawing it to my attention.

1. *Bending of a stretched semi-infinite beam*

The equation of moment equilibrium for the semi-infinite beam shown in Fig. 5(a) is

$$EI\frac{d^2w}{dx^2} = M - Nw \tag{B1}$$

where w is the deflection of the beam. The solution of this equation satisfying the boundary conditions $w = 0$ at $x = 0$, and w finite as $x \to \infty$ is

$$w = \frac{M}{N}(1 - e^{-\alpha x}), \tag{B2}$$

where $\alpha = (N/EI)^{1/2}$.

The displacement at infinity and the rotation θ at $x = 0$ are respectively

$$w_\infty = M/N \qquad \text{and} \qquad \theta = \alpha M/N. \tag{B3}$$

2. *Stretched beam with thickness discontinuity*

Consider now a beam which has a discontinuity in thickness from h on one side to H on the other as shown in Fig. 5(b). Let M_1 and M_2 be the bending moments induced at the discontinuity when a tension N is applied. For moment equilibrium we must have, taking M_1, M_2 positive in the directions shown in Fig. 5(b),

$$M_1 + M_2 = \tfrac{1}{2}(H - h)N. \tag{B4}$$

Since the rotation θ of each beam at the junction must be the same, we have, using the result (B3)

$$\theta = \alpha_1 M_1/N = \alpha_2 M_2/N, \tag{B5}$$

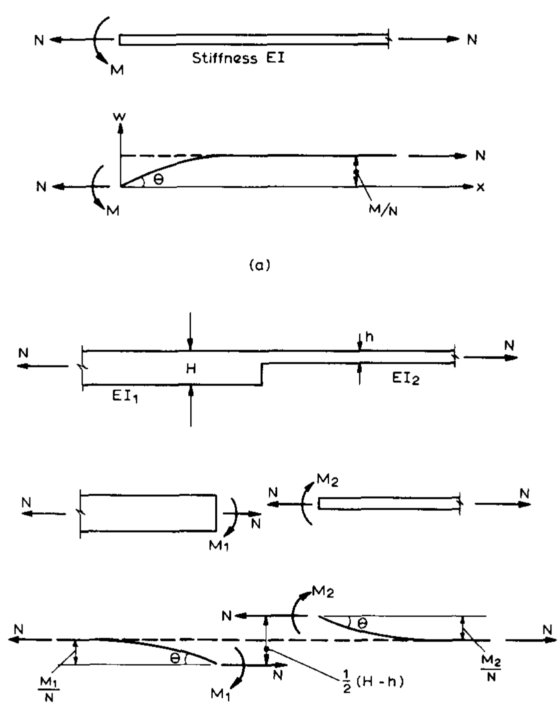

Fig. 5. (a) Bending of a stretched semi-infinite beam. (b) Stretched beam with thickness discontinuity.

where $\alpha_1 = (N/EI_1)^{1/2}$, $\alpha_2 = (N/EI_2)^{1/2}$, and we note that

$$\alpha_2/\alpha_1 = (I_1/I_2)^{1/2} = (H/h)^{3/2} = \lambda^{-3/2}, \tag{B6}$$

where $\lambda = h/H$ as previously.

Solving (B4) and (B5) for M_1 and M_2 and using (B6) we obtain

$$M_1 = (1+\lambda^{3/2})^{-1}\tfrac{1}{2}(H-h)N, \tag{B7}$$

$$M_2 = \lambda^{3/2}(1+\lambda^{3/2})^{-1}\tfrac{1}{2}(H-h)N, \tag{B8}$$

in agreement with (35a) and (34a) respectively. Note that $\tilde{M}_{rr}$(plate) $= -M_2$ because of the choice of positive direction for M_2. The deflections of the two beams at infinity are M_1/N and M_2/N as shown in Fig. 5(b), and since from (B4) $(M_1+M_2)/N = \frac{1}{2}(H-h)$, the deflections are such as to make the neutral axes of the two beams colinear at large distances from their junction.

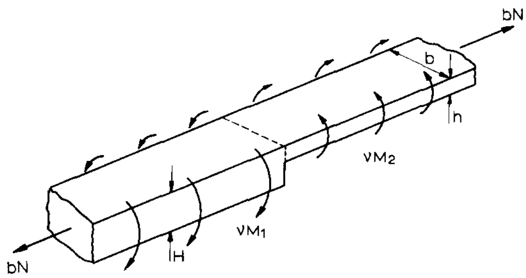

Fig. 6. Representative beam in the plate boundary layer.

3. *Application to the stretched plate with an eccentrically plate reinforced hole*

Consider a plate containing an eccentrically plate reinforced hole of arbitrary shape, subjected to large tension far from the hole, as shown in Fig. 2. In the asymptotic state it is reasonable to represent the behaviour in the boundary layers by considering a beam B whose axis is normal to the curve C of the junction between the plate and the reinforcement (Fig. 2).

Within the boundary layer the variation of tension in this beam and its variation in width, due to the divergence of the normals to C, can be ignored, provided the boundary layer is very narrow and the curvature of C is everywhere moderate. Thus, the beam B can be treated as a beam of uniform width, b say, in a state of uniform tension as shown in Fig. 6. The curvature across the width of the beam is zero in the first approximation and this requires transverse bending moments νM_1 and νM_2 as shown in Fig. 6. Thus the effective flexural stiffnesses of the two halves of the beam are

$$EbH^3[12(1-\nu^2)]^{-1} \quad \text{and} \quad Ebh^3[12(1-\nu^2)]^{-1}.$$

However, since E does not appear in equations (B7) and (B8), this effect is irrelevant and these equations apply at the junction C of the plate and reinforcement.